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THE EQUATIONS OF MOTION OF A NON-HOLONOMIC SYSTEM WITH A NON-RETAINING CONSTRAINT*

A.P. IVANOV

The regularity of the equations of motion of a system with a perfect non-retaining constraint $q_1 \geqslant 0$ and with differential constraints is demonstrated. As regards the latter it is assumed that they are imposed either on all motions of the system or only on those for which $q_1 = 0$. The effect of impacts on the stability of permanent rotation of a heavy solid about its axis of symmetry above an absolutely rough surface is investigated. It is shown that the stability of rotation of a solid on the surface can be destabilized by tearing away to an arbitary height, as small as desired.

The possibility of deriving the equations of motion in regular form which defines the motion of a holonomic system with non-retaining constraint in an arbitrary time interval was showing earlier /1/. The advantages of this approach in comparison with the traditional method of "fitting" were demonstrated in /2-4/.

1. Suppose we are given a mechanical system M, defined in the configuration space $q \in R^n$ by generalized forces $\mathbf{Q}$ and the kinetic energy T, which is a quadratic form in $\mathbf{q}^{\cdot}$. The motion of the system is restrained by a non-retaining constraint $q_1 \geqslant 0$, and by $m < n$ differentiable constraints of the form

$$c_i = a_i\mathbf{q}^{\cdot} = 0,\ a_i = \mathbf{a}_i(\mathbf{q}, t) \in R^n \quad (i = 1, \ldots, m) \tag{1.1}$$

We shall consider two types of differentiable constraints, assuming that only those motions for which $q_1 = 0$, and for $i = m_1 + 1, \ldots, m$ all motions of system M, obey relations (1.1).

If the coordinate q_1 vanishes when $t = t^*$ an impact occurs on the non-retaining constraint, as well as the differential constraints of the first type. According to Newton's hypothesis that impact (considered absolutely elastic) can be defined by the relations

$$q_1^{\cdot}(t^* + 0) = -q_1^{\cdot}(t^* - 0),\quad c_j(t^* + 0) = -c_j(t^* - 0) \tag{1.2}$$
$$(j = 1, \ldots, m_1)$$

We describe the motion free of impacts by the Boltzmann-Hamel Eqs./5/. If the quasicoordinates π are defined by a reversible substitution

$$\pi_i^{\cdot} = \mathbf{l}_i\mathbf{q}^{\cdot},\ \pi_{n-m+j}^{\cdot} = c_j,\ \mathbf{l}_i(\mathbf{q}, t) \in R^n \tag{1.3}$$
$$(i = 1, \ldots, n - m;\ j = 1, \ldots, m)$$

these equations have in region $q_1 > 0$ the form

$$\frac{d}{dt}\frac{\partial T}{\partial \pi_s^{\cdot}} - \frac{\partial T}{\partial \pi_s} + \gamma_{sij}\frac{\partial T}{\partial \pi_i^{\cdot}}\pi_j^{\cdot} = \Pi_s,\quad \pi_r^{\cdot} = 0 \tag{1.4}$$
$$(s = 1, \ldots, n - m + m_1;\ r = n - m + m_1 + 1, \ldots, n)$$

where the kinetic energy T is set up taking relations (1.3) into account, Π_s is the generalized force corresponding to the quasicoordinate π_s and the coefficients of non-holonomy γ are determined from the permutational relations

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$$(d\delta - \delta d)\,\pi_k = \gamma_{ikj}\,\delta\pi_i d\pi_j \quad (k = 1, \ldots, n) \tag{1.5}$$

In formulae (1.4) and (1.5) the summation signs over the indices i, j are omitted.

The motions of system M for which $q_1 \equiv 0$ may also be described using Eqs.(1.4) in which the active forces acting on the system must be supplemented by the reaction of a non-retaining constraint, and the indices must be changed in the limits $s = 1, \ldots, n-m$; $r = n-m+1, \ldots, n$.

To formalize the impact interactions it is convenient to set

$$\pi_1 = q_1, \quad \pi_i^{\cdot} = \partial T'/\partial q_i^{\cdot} \quad (i = 2, \ldots, n-m) \tag{1.6}$$

in (1.3), where T' is the kinetic energy whose composition must take into account the second group of relations (1.3). The quasivelocities after the impact are then determined by conditions (1.2), and the property of continuity $\pi_i^{\cdot}$ $(i = 2, \ldots, n-m)$ proved in /6/. These values may be considered as the initial conditions for system (1.4) that determines the motions in the time interval to the second impact, and so forth. Obtaining such conclusions on the properties of motion in an infinite time interval is, however, difficult using this method, called the method of fitting, due to the lack of a priori information on the instants of impact.

The other method of investigation is to describe the system M using some ancilliary system M^* not subject to impacts. The advantages of this approach were demonstrated in /1-4/ for holonomic systems. A development of this method for systems with constraints of the form (1.1) is proposed below.

The motion of system M^* will be described in the phase space $(\mathbf{q}^*, \boldsymbol{\pi}^{*\cdot})$ by setting the correspondence between the trajectories of M and M^* using relations.

$$q_1 = |q_1^*|, \quad q_2 = q_2^*, \ldots, q_n = q_n^*, \quad \pi_i^{\cdot} = \pi_i^{*\cdot}(\operatorname{sgn} q_1^*)^{\nu_i} \tag{1.7}$$
$$\nu_i = \begin{cases} 1 \text{ when } i = 1,\, n-m+1, \ldots, n-m+m_1 \\ 0 \text{ otherwise} \end{cases}$$

Theorem. In phase space the trajectories (1.7) are defined by the system of equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial \pi_s^{*\cdot}} - \frac{\partial T^*}{\partial \pi_s^*} + \gamma_{sij}^* \frac{\partial T^*}{\partial \pi_i^{*\cdot}}\pi_j^{*\cdot} = \Pi_s^*, \quad \pi_r^{*\cdot} = 0 \tag{1.8}$$
$$(s = 1, \ldots, n-m+m_1;\ r = n-m+m_1+1, \ldots, n)$$
$$\gamma_{sij}^* = \gamma_{sij}(\operatorname{sgn} q_1^*)^{\nu_s+\nu_i+\nu_j}, \quad \Pi_s^* = \Pi_s(\operatorname{sgn} q_1^*)^{\nu_s}$$

where T^* is the kinetic energy in Eqs.(1.4) in which expressions (1.7) are substituted for q, $\pi^{\cdot}$.

Lemma. Let $N = {}^1/_2\boldsymbol{\alpha}\mathbf{A}\boldsymbol{\alpha}^T$ be a non-degenerate quadratic form, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. The variables $\boldsymbol{\beta}$ are connected with $\boldsymbol{\alpha}$ by the relations

$$\beta_r = \partial N/\partial \alpha_r, \ \beta_s = \alpha_s \ (r = 1, \ldots, m;\ s = m+1, \ldots, n)$$

in which in the expression $N = {}^1/_2\boldsymbol{\beta}\mathbf{B}\boldsymbol{\beta}^T$ there are no products of different groups; $b_{rs} = 0$.

Proof of the lemma. We divide matrix $\mathbf{A}$ into blocks, separating m of its rows and m columns

$$\mathbf{A} = \begin{Vmatrix} \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{A}_1^T & \mathbf{A}_2 \end{Vmatrix}, \quad \mathbf{A}_0^T = \mathbf{A}_0, \quad \mathbf{A}_2^T = \mathbf{A}_2$$

We will represent the relation between α and β in the form

$$\beta = \mathbf{L}\alpha, \quad \mathbf{L} = \begin{Vmatrix} \mathbf{A}_0 & \mathbf{A}_1 \\ 0 & \mathbf{E} \end{Vmatrix}$$

where $\mathbf{E}$ is the unit matrix of order $n-m$.

For matrix $\mathbf{B}$ we obtain

$$\mathbf{B} = (\mathbf{L}^{-1})^T\mathbf{A}\mathbf{L}^{-1} = \mathbf{A}_0^{-1} \oplus (\mathbf{A}_2 - \mathbf{A}_1^T\mathbf{A}_0^{-1}\mathbf{A}_1)$$

from which the above statement follows.

Proof of the theorem. By the lemma, the kinetic energy T^* decomposes into the sum of two quadratic forms, each of which corresponds to a definite value of ν_i, calculated for the indices of the variables appearing in it. Hence the factors $\operatorname{sgn} q_1^*$ will not appear in T^*, and system (1.8) will not contain singularities of the delta-function type. Consequently, the solutions of this system are represented by continuous curves in $(\mathbf{q}^*, \boldsymbol{\pi}^{*\cdot})$ space. When $q_1^* \geqslant 0$ the congruence of (1.4) and (1.8) indicates that such solutions are identical with the solutions of system (1.4). The passage of the trajectory of system M^* through the plane $q_1^* = 0$ ensures, by virtue of (1.7), that the conditions of impact (1.2) for system M are satisfied.

It can be shown that when $q_1^* < 0$, the values of γ^* defined by (1.8) are identical with those that they take by virtue of (1.5), formulated for π^*, and the generalized forces Π^* have values determined by the power-balance equations.

$$\delta A = \sum_{i=1}^{n} \Pi_i \delta\pi_i = \sum_{i=1}^{n} \Pi_i^* \delta\pi_i^*$$

From this follows the symmetry of the solutions of system (1.4) and (1.8) as expressed by (1.7).

As an example we shall consider a heavy uniform sphere of unit mass, weight, and radius, rolling without slip on an inclined plane P_1, and colliding with an absolutely rough surface P_2 perpendicular to P_1. The plane P_2 in this formulation represents the non-retaining constraint and the differential constraint of the first type, while the plane P_1 represents the two differential constraints of the second type.

We introduce the inertial system of coordinates $OXYZ$ with semi-axes OX and OZ in planes P_1 and P_2 normal to the axis OY directed along the line of intersection of P_1 and P_2 so as to have a right-hand system of coordinates. As the Lagrange coordinates we take $q_1 = x - 1,\ q_2 = y,\ q_3 = \theta,\ q_4 = \psi,\ q_5 = \varphi$, where x, y are the coordinates of the centre of the sphere $(z \equiv 1)$, and θ, ψ, φ are the Euler angles.

The kinetic energy T, the generalized forces Q, and the constraints imposed on the system have the form

$$\begin{aligned} &T = {}^1/_2\,(q_1^{\cdot 2} + q_2^{\cdot 2}) + {}^1/_2 a^2\ (\omega_x^2 + \omega_y^2 + \omega_z^2) \\ &Q_1 = -\cos\alpha,\ Q_2 = -\cos\beta,\ Q_3 = Q_4 = Q_5 = 0 \\ &q_1 \geqslant 0,\ c_1 = q_2^{\cdot} - \omega_z = 0,\ c_2 = q_1^{\cdot} - \omega_y = 0,\ c_3 = q_2^{\cdot} + \omega_x = 0 \\ &\omega_x = \theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\theta\sin\psi,\quad \omega_y = \theta^{\cdot}\sin\psi - \varphi^{\cdot}\sin\theta\cos\psi,\quad \omega_z = \\ &\quad \varphi^{\cdot}\cos\theta + \psi^{\cdot} \end{aligned} \tag{1.9}$$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the coordinates of the unit vector directed vertically upwards, ω_x ω_y, ω_z are the coordinates of the instantaneous angular velocity vector ω, and a is the radius of inertia of the sphere relative to its diameter. The constraints $c_1 = 0$ and $c_2 = 0$ $c_3 = 0$ are constraints of the first and second type, respectively.

The quasicoordinates π, in conformity with (1.3) and (1.6), have the form

$$\begin{aligned} &\pi_1 = q_1,\ \pi_3^{\cdot} = c_1,\ \pi_4^{\cdot} = c_2 \equiv 0,\ \pi_5^{\cdot} = c_3 \equiv 0 \\ &\pi_2^{\cdot} = \partial T'/\partial q_2^{\cdot} = (1 + 2a^2)\, q_2^{\cdot} - a^2 \pi_3^{\cdot} \end{aligned}$$

The equations of motion for $q_1 > 0$ have the form (1.4), where

$$\begin{aligned} &2T = (1 + a^2)\,\pi_1^{\cdot 2} + \frac{\pi_2^{\cdot 2} + a^2(1 + a^2)\,\pi_3^{\cdot 2}}{1 + 2a^2} \\ &\Pi_1 = Q_1,\ \Pi_2 = \frac{Q_2}{1 + 2a^3},\quad \Pi_3 = \frac{a^2}{1 + 2a^2} Q_2 \\ &\gamma_{231} = -\gamma_{132} = \gamma_{243} = -\gamma_{342} = \gamma_{152} = -\gamma_{251} = \frac{-1}{1 + 2a^2} \\ &-\gamma_{122} = +\gamma_{221} = -\gamma_{331} = +\gamma_{133} = \frac{a^2}{1 + 2a^2} \\ &-\gamma_{123} = +\gamma_{321} = \frac{a^4}{1 + 2a^2},\quad -\gamma_{351} = +\gamma_{153} = \frac{1 + a^2}{1 + 2a^2} \end{aligned}$$

and the remaining non-holonomic coefficients γ are zero.

Eqs.(1.8) that define trajectories of system M^* have the form

$$\begin{aligned} &\dot{\pi}_1^{\cdot *} = -\frac{\cos\alpha}{1 + a^2}\operatorname{sgn} q_1^*,\quad \dot{\pi}_2^{\cdot *} = -\cos\beta \\ &\dot{\pi}_3^{\cdot *} = -\frac{\cos\beta}{1 + a^2}\operatorname{sgn} q_1^*,\quad \dot{\pi}_4^{*} = 0,\quad \dot{\pi}_5^{*} = 0 \end{aligned} \tag{1.10}$$

Assuming that $\cos\alpha > 0,\ q_1^*(0) = 0$ and integrating system (1.10), we obtain

$$\begin{aligned} &q_1^* = -\frac{\cos\alpha}{2(1 + a^2)}\, t\,(|t| - \tau),\quad -\tau \leqslant t \leqslant \tau \\ &q_1^*(t + 2\tau) = q_1^*(t),\ \pi_2^{\cdot} = \pi_2^{\cdot *} = A_1 - t\cos\beta \\ &\pi_3^{\cdot} = \dot{\pi}_3^{*}\operatorname{sgn} q_1^* = -\frac{\cos\beta}{1 + a^2}\, t + A_2 \operatorname{sgn} q_1^* \\ &q_2^{\cdot} = \frac{A_1 + a^2 A_2 \operatorname{sgn} q_1^*}{1 + 2a^2} - \frac{\cos\beta}{1 + a^2}\, t \\ &\omega_x = -q_2^{\cdot},\quad \omega_y = q_1^{\cdot},\quad \omega_z = \frac{A_1 - (1 + a^2) A_2 \operatorname{sgn} q_1^*}{1 + 2a^2} \end{aligned} \tag{1.11}$$

where the constants τ, A_1, A_2 are determined by the initial conditions.

It follows from (1.11) that the trace of the sphere on the plane P_1 is a broken line consisting of parabolic arcs, with q_1 and ω_y periodically changing, and the quantity ω_z

takes in turn one of two constant values.

2. Let us consider the motion of a heavy axisymmetric solid above an absolutely rough horizontal plane P. The condition of no-slip of the solid over the plane imposes on the system two differential constraints of the first type. The expressions for these obtained in /7/ may be written as follows:

$$v_1 = f\omega_2 + f'\omega_3 \sin\alpha,\quad v_2 = f\omega_1 - f'\omega_3\cos\alpha \tag{2.1}$$

where $f = f(\beta) = GG'$ is the distance from the centre of mass G to the plane P, measured at the contact, α and β are respectively the longitude and polar distance of the axis of the solid GZ' relative to the frame of reference $Ge_1e_2e_3$, $e_3 \perp P$ moving translationally (Fig. 1), and v_1, v_2, v_3 and $\omega_1, \omega_2, \omega_3$ are the projections of the velocity of the point G and of the instantaneous velocity of the solid on the directions of the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

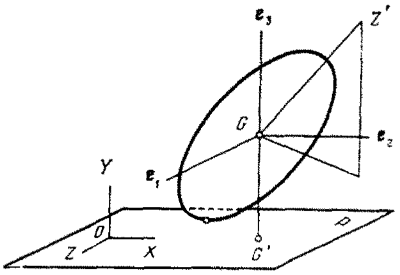

Fig.1

We select the origin of the inertial system of coordinates $OXYZ$ in the plane P, and direct $OX \parallel \mathbf{e}_2, OY \parallel \mathbf{e}_3, OZ \parallel \mathbf{e}_1$. The Lagrange function L and the superposed non-retaining and differential constraints have the form

$$L = {}^1/_2 m[x^{\cdot 2} + (q_1^{\cdot} + f'\beta^{\cdot})^2 + z^{\cdot 2}] + {}^1/_2 A(\theta^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta) + {}^1/_2 C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2 - mg(q_1 + f),\quad \cos\beta = -\sin\theta\cos\psi \tag{2.2}$$

$$q_1 = y - f(\beta) \geqslant 0,\quad c_1 = x^{\cdot} + f\omega_z + \frac{f'}{\sin\beta}\omega_Y\cos\theta \tag{2.3}$$
$$c_2 = z^{\cdot} - f\omega_X - \frac{f'}{\sin\beta}\omega_Y\sin\theta\sin\psi,\quad c_1 = c_2 = 0 \text{ for } q_1 = 0$$
$$\omega_X = \theta^{\cdot}\cos\psi + \psi^{\cdot}\sin\theta\sin\psi,\quad \omega_Y = \theta^{\cdot}\sin\psi - \psi^{\cdot}\sin\theta\cos\psi$$
$$\omega_Z = \varphi^{\cdot}\cos\theta + \psi^{\cdot}$$

where x, y, z are the coordinates of the point G, θ, ψ, φ are the Euler angles, m and mg are the mass and weight of the solid, and A and C are the equatorial and axial moments of inertia.

We determine the quasicoordinates π, in conformity with (1.3) and (1.6)

$$\pi_1 = q_1,\quad \pi_5^{\cdot} = c_1,\quad \pi_6^{\cdot} = c_2,\quad \pi_{2,3,4}^{\cdot} = \frac{\partial L'}{\partial\theta^{\cdot}, \psi^{\cdot}, \varphi^{\cdot}} \tag{2.4}$$

where L' is a function of L in which $x^{\cdot}, z^{\cdot}$ are eliminated using relations (2.3).

The equations of motion may be written in the form (1.8). As can be shown, they have particular solutions of the form

$$q_1^* = -{}^1/_2 gt(|t| - T) \text{ when } -T \leqslant t \leqslant T,\quad q_1^*(t + 2T) = q_1^*(t) \tag{2.5}$$
$$\theta = {}^1/_2\pi,\quad \pi_2^{\cdot *} = 0,\quad \psi = \pi,\quad \pi_3^{\cdot *} = 0,\quad \varphi^{\cdot} = \Omega,\quad \pi_4^{\cdot *} = C\Omega,$$
$$\pi_{5,6}^{\cdot *} = 0$$

that correspond to permanent rotations about the axis of symmetry situated vertically, for periodic impacts with the constraining plane.

To a first approximation of the stability of the vertical orientation of the axis of the solid we must assume that $q_1^*, \varphi, \pi_4^{\cdot *}$ in the perturbed motion have the same form (2.5) /8/. The characteristic equation for a system in variations of the quantities $\theta, \psi, \pi_2^{\cdot *}, \pi_3^{\cdot *}, \pi_5^{\cdot *}, \pi_6^{\cdot *}$ may be composed, taking into account (2.4), the continuity of the quasivelocities $\pi_i^{\cdot *}$ $(i = 1, \ldots, 6)$ on impact, and the relations

$$x^{\cdot\cdot} = 0,\quad z^{\cdot\cdot} = 0,\quad A\theta^{\cdot\cdot} + C\Omega\psi^{\cdot} = 0,\quad A\psi^{\cdot\cdot} - C\Omega\theta^{\cdot} = 0,$$
$$\varphi^{\cdot} = \Omega$$

which defines the motion of the solid in the intervals between the impacts on the plane.

That equation is reciprocal; using the change $w = {}^1/_2(\rho + \rho^{-1})$ it can be reduced to the form

$$\chi(\omega) = w^3 + a_2w^2 + a_1w + a_0 = 0 \tag{2.6}$$
$$a_2 = -1 + u_1s_0 + (1 - c_0)(1 - u_2),\quad \alpha_1 = -1 + {}^1/_2(1 - c_0)\times (u_1^2 + u_2^2)$$
$$a_0 = 1 + {}^1/_2(1 - c_0)[u_1^2 - 1 - (1 - u_2)^2] - u_1s_0$$
$$s_0 = \sin C/A\,\Omega T$$
$$c_0 = \cos\frac{C}{A}\Omega T,\quad u_1 = \frac{mf''(0)\,TA}{[A + mf'^2(0)]\,C\Omega}$$
$$u_2 = \left[1 - \frac{Ar}{Cf(0)}\right]\frac{mf^2(0)}{A + mf^2(0)}$$

where $r = f(0) + f''(0)$ is the radius of curvature of the surface of the solid at the point of its axis.

For solution (2.5) to be stable it is necessary for (2.6) to have three roots in the interval $[-1, 1]$. This is equivalent to the system of inequalities

$$|a_2| \leqslant 3,\ 2|a_2| \leqslant 3 + a_1,\ |a_0 + a_2| \leqslant 1 + a_1$$
$$27a_0^2 - 18a_0a_1a_2 + 4a_0a_2^3 - 4a_1^3 - a_1^2a_2^2 \leqslant 0 \tag{2.7}$$

Let us consider some special cases of the motion.

1°. Point G coincides with the centre of curvature of the surface on the axis of the solid, i.e. $r = f(0)$. Then $u_1 = 0$, and one of the roots (2.6) is unity and system (2.7) is equivalent to two inequalities

$$1 - c_0 \leqslant \frac{2}{(1 - u_2)^2},\quad 1 - c_0 \leqslant \frac{2}{|1 - u_2|} \tag{2.8}$$

The analysis of conditions (2.8) shows that when $A < C$, they are satisfied for any T, Ω. If, however, $A > C$ (an elongated solid), then in the parameter plane T, Ω there are zones lying near the curves

$$C\Omega T = A\pi(1 + 2N)\ (N = 0, \pm 1, \pm 2, \ldots)$$

that correspond to instability of the solutions (2.5). Note that similar motions in the case of a smooth surface are stable /8, 9/.

2°. In steady motion $\Omega = 0$. Then the characteristic equation (2.6) has the form

$$(w + 1)\left(w + \frac{u^*}{2} - 1\right)^2 = 0,\quad u^* = \frac{mgf''(0)T^2}{A + mf^2(0)}$$

and the necessary condition of stability is

$$0 \leqslant u^* \leqslant 2 \tag{2.9}$$

Conditions (2.9) are somewhat less rigid compared with such conditions for motions of the solid on a smooth surface /8, 9/.

3°. The velocity of rotation is large compared with that of vertical translations $u_1 \ll 1$ of the body. It can be shown that if inequalities (2.8) are satisfied in the strict sense for some values of u_2 and c_0, then for fairly small u_1 (and fixed u_2 and c_0) the inequalities (2.7) are satisfied, and solutions (2.5) are, to a first approximation, stable. However, it appears that the maximum of u_1 for which the stability is not violated, decreases to zero as $c_0 \to 1$.

Indeed, the quantity

$$\chi'(1) = 2u_1s_0 + {}^1/_2(1 - c_0)[u_1^2 + (u_2 - 2)^2] =$$
$${}^1/_2(1 - c_0)\ [u_1^2 + (u_2 - 2)^2 - 4u_1 \operatorname{ctg}({}^1/_2 C\Omega TA^{-1})]$$

is negative for values of ${}^1/_2C\Omega TA^{-1}$ that are close to, but somewhat smaller, than an integer multiple of π. Hence it follows that the derivative $\chi'(w)$ has a root greater than unity, while in the case of stability all its roots necessarily belong to the interval /-1, 1/.

Let us compare this form of stability of solutions (2.5) when $u_1 \ll 1$ with the stability condition of permanent rotation of a solid permanently touching the supporting plane, that has the form /10/

$$[C + mf(0)r]^2\Omega^2 \geqslant 4f''(0)mg[A + mf^2(0)] \tag{2.10}$$

Relation (2.10) is satisfied if Ω is fairly large, and in case when $f''(0) \leqslant 0$ for any Ω. It follows from the above that the assumption that it is possible to separate the solid from the supporting plane for an arbitrarily small height, results, for specific values of Ω, in destabilization of stable motions.

On the other hand, conditions (2.10) may not be satisfied, but the inequalities (2.8) ensure stability to a first approximation when $u_1 \neq 0$ is fairly small.

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THE PROBLEM OF CONSTRUCTING A LYAPUNOV FUNCTION *

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An algorithm which, for a wide class of problems, enables a Lyapunov function with a negative-sign derivative to be reconstructed as a Lyapunov function with a negative-definite derivative, is proposed. This algorithm supplements the well-known method /1/ of reconstructing a Lyapunov function. Examples are considered.

Consider a set of differential equations of perturbed motion

$$x_i^{\cdot} = f_i(x),\ f(0) = 0,\ x \in R^n,\ f_i(x) \in C^1(\Omega),\ \{0\} \in \Omega \subset R^n \tag{1}$$

We will assume that for (1) Lyapunov's function $V_0(x)$, which is positive definite in the domain Ω and whose time-derivative is non-positive in this domain and vanishes in the manifold $M \subset \Omega$ by virtue of Eqs.(1), is known.

We shall formulate the problem of determining the functions $V_\nu(x)$ $(\nu \leqslant n-1)$ and the constants $\mu_\nu > 0$, for which the sum

$$V(x) = V_0(x) + \sum_{\nu=1}^{p} \mu_\nu V_\nu(x), \quad p \leqslant n-1 \tag{2}$$

(the quantity p is refined while solving the problem) will be positive definite, and its time derivative is, by virtue of (1), a negative-definite function in Ω.

We shall show that for the additional assumptions introduced below this problem has the following solution.

Suppose the manifold M is described by the equations $S_1(x) = 0, \ldots, S_m(x) = 0$, which are solvable in Ω with respect to certain m variables, for example

$$x_j = x_j^\circ(x_{m+1}, \ldots, x_n),\ x_j^\circ(0) = 0,\ j = 1, \ldots, m$$

We shall determine the functions f_i° and Φ_k $(i, k = 1, \ldots, n)$ using the equations

$$f_i^\circ(x_{m+1}, \ldots, x_n) = f_i(x_1^\circ(x_{m+1}, \ldots, x_n), \ldots, x_m^\circ(x_{m+1}, \ldots, x_n), x_{m+1}, \ldots, x_n) \tag{3}$$

$$\Phi_k(x_k, x_{m+1}, \ldots, x_n) = -\int_0^{x_k} f_k^\circ(x_{m+1}, \ldots, x_n)\, dx_k + \Phi_{0k} \tag{4}$$

Here Φ_{0k} is an arbitrary function of the coordinates $x_{m+1}, \ldots, x_n$, in a number of which the coordinate x_k does not occur, and $\Phi_{0k}(0) = 0$. (When $k \geqslant m+1$ x_k is omitted in the left-hand side of (4)).

If the functions f_i do not depend on $x_{m+1}, \ldots, x_n$, we will assume that $f_i^\circ \equiv 0$.

We shall determine the function $V_{*1}(x)$ in the form of the sum

$$V_{*1}(x) = \sum_{k=1}^{n} \lambda_k \Phi_k(x)$$

in which the constants λ_k will be determined below.

We shall write the time-derivative of this function by virtue of (1)

$$V_{*1}^{\cdot}(x) = \sum_{i=1}^{n} f_i(x) \sum_{k=1}^{n} \lambda_k \frac{\partial \Phi_k(x)}{\partial x_i}$$

Since for $i \leqslant m$

*Prikl.Matem.Mekhan.,49,5,724-729,1985